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Useful tools for non-linear systems: Several non-linear integral inequalities

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ABSTRACT

Integral inequalities play important roles in classical probability and measure theory. Universal integrals provide a useful tool in many problems in engineering and non-linear systems where the aggregation of data is required. We discuss several inequalities including Hardy, Berwald, Barnes–Godunova–Levin, Markov and Chebyshev for a monotone measure-based universal integral. Some recent results are obtained as corollaries. Finally, we provide some applications of our results in intelligent decision support systems, estimation and information fusion.

1. Introduction

The theory of non-additive measures was first developed by Sugeno [42] as a tool for modeling nondeterministic problems. There are considered several important kinds of non-additive integrals. One of them is Sugeno integral which has been generalized by using some other operators to replace the special operator(s) $\wedge$ and/or $\vee$ (see, e.g., [29,31,41,43]).

Choquet integral is another important kind of non-additive integrals which was first introduced by Choquet [9] and has been studied by many other researchers [12,34].

The Choquet and the Sugeno integral provide a useful tool in many problems in engineering, non-linear systems and fuzzy inference systems where the aggregation of data is required [44]. However, their applicability is restricted because of the special operations used in the construction of these integrals. Therefore, Klement et al. [24,21,22] provided a universal integral generalizing both the Choquet and the Sugeno case.

The integral inequalities are useful tools in several theoretical and applied fields. They are a part of the classical mathematical analysis [17,26]. For the case of the classical Riemann integral, we recall some well-known integral inequalities and we aim to generalize these inequalities for the universal integrals.

Definition 1.1.

(i) Let $f:[0,\infty)\to[0,\infty)$ be an integrable function ($f\neq 0$) and $F(x)=\int_0^x f(t)dt$. Then the classical Hardy type inequality holds:

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx,$$

where $p > 1$.

(ii) Let f be a nonnegative concave function on $[a,b]$. Then the classical Berwald type inequality holds [36]:

$$\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\int_a^b f^s(x)dx}{b-a}\right)^{\frac{1}{s}} \leqslant \left(\frac{\int_a^b f^r(x)dx}{b-a}\right)^{\frac{1}{r}},$$

for $0 < r < s < \infty$

(iii) The following is the classical Barnes–Godunova–Levin inequality [36]:

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leqslant B(p,q)\int_a^b f(x)g(x)dx,$$

where $p,q>1, B(p,q)=\frac{6(b-a)^{\frac{1}{p}+\frac{1}{q}-1}}{(1+p)^{\frac{1}{p}}(1+q)^{\frac{1}{q}}}$ and f, g are nonnegative concave functions on $[a,b]$.

(iv) Let f be a nonnegative function. Then the classical Markov type inequality holds:

$$\frac{1}{c}\int_A f d\mu \geqslant \mu\{x\in A | f(x) \geqslant c\},$$

for $c > 0$.

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The study of inequalities for Sugeno integral was initiated by Romá n-Flores et al. [14,37], and then followed by many authors [1–5,8,14,30,33,37]. Recently, Román-Flores et al. [37] proved a Hardy type inequality for a measurable function and Lebesgue measure-based Sugeno integral, and then Agahi and Yaghoobi [5] further generalized it to comonotone functions. In [14], Markov type inequalities for Sugeno integral were obtained which has been generalized by Caballero and Sadarangani [8]. Furthermore, the classical Berwald inequality for monotone functions and Lebesgue measure-based Sugeno integral was proposed in a special form by Agahi et al. [2]. Also, Agahi et al. [1] proved a strengthened version of Barnes–Godunova–Levin type inequality for Sugeno integrals on a real interval based on a binary operation.

The aim of this contribution is to generalize their works to the frame of the universal integral on abstract spaces. In fact, this paper generalizes most of previous results [1,2,5,14,30,33,37].

To stress the applicability of non-linear integrals in the area of knowledge-based systems, we recall few references: in [32] such integrals were added to fuzzy control systems; description of some non-linear integrals and their use in information theory can be found in the monograph [25]; for applications of the Choquet integral in tactical knowledge representation see [38]; typicality analysis and feature selection based on Choquet integral is the topic of [28]; overtimes strategies based on non-linear integrals are discussed in [11].

The paper is organized as follows. In the next section, we briefly recall some preliminaries and summarization of some previous known results. In Section 3, we focus on several inequalities including Hardy, Berwald, Markov and Chebyshev for universal integral. In Section 4, we provide some applications of our results in intelligent decision support systems, estimations and information fusion. Finally, a conclusion is given.

2. Universal integral

In this section, we are going to review some well known results from universal integral (see [24]).

Definition 2.1 [24]. A monotone measure m on a measurable space $(X,\mathcal{A})$ is a function $m:\mathcal{A}\to[0,\infty]$ satisfying

(i) $m(\phi)=0$,
(ii) $m(X)>0$,
(iii) $m(A)\leqslant m(B)$ whenever $A\subseteq B$.

Normed monotone measures on $(X,\mathcal{A})$, i.e., monotone measures satisfying $m(X)=1$, are also called capacities [15,42,45], depending on the context.

For a measurable space $(X,\mathcal{A})$, i.e., a non-empty set X equipped with a σ-algebra $\mathcal{A}$, recall that a function $f{:}X\to[0,\infty]$ is called $\mathcal{A}$-measurable if, for each $B\in\mathcal{B}([0,\infty])$, the σ-algebra of Borel subsets of $[0,\infty]$, the preimage $f^{-1}(B)$ is an element of $\mathcal{A}$.

Definition 2.2 [24]. Let $(X,\mathcal{A})$ be a measurable space.

(i) $\mathcal{F}^{(X,\mathcal{A})}$ denotes the set of all $\mathcal{A}$-measurable functions $f{:}X\to[0,\infty)$;
(ii) For each number $a\in(0,\infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ denotes the set of all monotone measures (in the sense of Definition 2.1) satisfying $m(X)=a$; and we take

$$\mathcal{M}^{(X,\mathcal{A})}=\bigcup_{a\in(0,\infty]}\mathcal{M}_a^{(X,\mathcal{A})}.$$

Let $\mathcal{S}$ be the class of all measurable spaces, and take

$$\mathcal{D}_{[0,\infty]}=\bigcup_{(X,\mathcal{A})\in\mathcal{S}}\mathcal{M}^{(X,\mathcal{A})}\times\mathcal{F}^{(X,\mathcal{A})}.$$

Definition 2.3. The Choquet [9], Sugeno [42] and Shilkret [40] integrals (see also [7,29,34,35]), respectively, are given, for any measurable space $(X,\mathcal{A})$, for any measurable function $f\in\mathcal{F}^{(X,\mathcal{A})}$ and for any monotone measure $m\in\mathcal{M}^{(X,\mathcal{A})}$, i.e., for any $(m,f)\in\mathcal{D}_{[0,\infty]}$, by

$$\mathbf{Ch}(m,f)=\int_0^\infty m(\{f\geqslant t\})dt,$$
$$\mathbf{Su}(m,f)=\sup\{\min(t,m(\{f\geqslant t\}))|t\in(0,\infty]\},$$
$$\mathbf{Sh}(m,f)=\sup\{t.m(\{f\geqslant t\})|t\in(0,\infty]\},$$

where the convention $0.\infty=0$ is used. All these integrals map $\mathcal{M}^{(X,\mathcal{A})}\times\mathcal{F}^{(X,\mathcal{A})}$ into $[0,\infty]$ independently of $(X,\mathcal{A})$. We remark that fixing an arbitrary $m\in\mathcal{M}^{(X,\mathcal{A})}$, they are non-decreasing functions from $\mathcal{F}^{(X,\mathcal{A})}$ into $[0,\infty]$, and fixing an arbitrary $f\in\mathcal{F}^{(X,\mathcal{A})}$, they are non-decreasing functions from $\mathcal{M}^{(X,\mathcal{A})}$ into $[0,\infty]$.

Definition 2.4 [24]. Two pairs $(m_1,f_1)\in\mathcal{M}^{(X_1,\mathcal{A}_1)}\times\mathcal{F}^{(X_1,\mathcal{A}_1)}$ and $(m_2,f_2)\in\mathcal{M}^{(X_2,\mathcal{A}_2)}\times\mathcal{F}^{(X_2,\mathcal{A}_2)}$ satisfying

$$m_1(\{f_1\geqslant t\})=m_2(\{f_2\geqslant t\})\quad\text{for all } t\in(0,\infty],$$

will be called integral equivalent, in symbols

$$(m_1,f_1)\sim(m_2,f_2).$$

Definition 2.5. [34,43]. A function $\otimes{:}[0,\infty]^2\to[0,\infty]$ is called a pseudo-multiplication if it satisfies the following properties:

(i) it is non-decreasing in each component, i.e., for all a_1, a_2, b_1, $b_2\in[0,\infty]$ with $a_1\leqslant a_2$ and $b_1\leqslant b_2$ we have $a_1\otimes b_1\leqslant a_2\otimes b_2$;
(ii) 0 is an annihilator of $\otimes$, i.e., for all $a\in[0,\infty]$ we have $a\otimes 0=0\otimes a=0$;
(iii) $\otimes$ has a neutral element different from 0, i.e., there exists an element $e\in(0,\infty]$ such that, for all $a\in[0,\infty]$, we have $a\otimes e=e\otimes a=a$.

Definition 2.6. For a given pseudo-multiplication on $[0,\infty]$, we suppose the existence of a pseudo-addition $\oplus$: $[0,\infty]^2\to[0,\infty]$ which is continuous, associative, non-decreasing and has 0 as neutral element (then the commutativity of $\oplus$follows, see [23]), and which is left-distributive with respect to $\otimes$i.e., for all a, b, $c\in[0,\infty]$ we have $(a\oplus b)\otimes c=(a\oplus c)\otimes(b\oplus c)$. The pair $(\oplus,\otimes)$ is then called an integral operation pair, see [7,24].

Definition 2.7 [24]. A function $\mathbf{I}:\mathcal{D}_{[0,\infty]}\to[0,\infty]$ is called a universal integral if the following axioms hold:

(I1) For any measurable space $(X,\mathcal{A})$, the restriction of the function $\mathbf{I}$ to $\mathcal{M}^{(X,\mathcal{A})}\times\mathcal{F}^{(X,\mathcal{A})}$ is non-decreasing in each coordinate;
(I2) there exists a pseudo-multiplication $\otimes$: $[0,\infty]^2\to[0,\infty]$ such that for all pairs $(m,c.\mathbf{1}_A)\in\mathcal{D}_{[0,\infty]}$

$$\mathbf{I}(m,c.\mathbf{1}_A)=c\otimes m(A);$$

(I3) for all integral equivalent pairs $(m_1,f_1),(m_2,f_2)\in\mathcal{D}_{[0,\infty]}$ we have $\mathbf{I}\,(m_1,f_1)=\mathbf{I}(m_2,f_2)$.

Theorem 2.8 [24]. *Let $\otimes:[0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$. Then the smallest universal integral $\mathbf{I}$ and the greatest universal integral $\mathbf{I}$ based on $\otimes$ are given by*

$$\mathbf{I}_{\otimes}(m,f) = \sup\{t \otimes m(X \cap \{f \geqslant t\}) | t \in (0,\infty])\},$$
$$\mathbf{I}^{\otimes}(m,f) = essup_m f \otimes \sup\{m(X \cap \{f \geqslant t\}) | t \in (0,\infty])\},$$

where $essup_m f = \sup\{t \in [0,\infty] | m(X \cap \{f \geqslant t\}) > 0\}$.

Remark 2.9. Clearly, $\mathbf{Su} = \mathbf{I}_{Min}$ and $\mathbf{Sh} = \mathbf{I}_{Prod}$, where the pseudo-multiplications *Min* and *Prod* are given (as usual) by $Min(a,b) = min(a,b)$ and $Prod(a,b) = a \cdot b$.

Remark 2.10 [24]. There is neither a smallest nor a greatest pseudo-multiplication on $[0,\infty]$. But, if we fix the neutral element $e \in (0,\infty]$, then the smallest pseudo-multiplication $\otimes_e$ and the greatest pseudo-multiplication $\otimes^e$ with neutral element $e$are given by

$$a \otimes_e b = \begin{cases} 0 & \text{if } (a,b) \in [0,e)^2, \\ \max(a,b) & \text{if } (a,b) \in [e,\infty]^2, \\ \min(a,b) & \text{otherwise}, \end{cases}$$

and

$$a \otimes^e b = \begin{cases} \min(a,b) & \text{if } \min(a,b) = 0 \text{ or } (a,b) \in (0,e]^2, \\ \infty & \text{if } (a,b) \in (e,\infty]^2, \\ \max(a,b) & \text{otherwise}. \end{cases}$$

Proposition 2.11 [24]. *There exists the smallest universal integral $\mathbf{I}_{\otimes_e}$ among all universal integrals satisfying the conditions*

(i) *for each* $m \in \mathcal{M}_e^{(X,\mathcal{A})}$ and each $c \in [0,\infty]$ we have $\mathbf{I}(m, c.\mathbf{1}_X) = c$,
(ii) *for each* $m \in \mathcal{M}^{(X,\mathcal{A})}$ and each $A \in \mathcal{A}$ we have $\mathbf{I}(m, e.\mathbf{1}_X) = m(A)$, *given by*

$$\mathbf{I}_{\otimes_e}(m,f) = \max\{m(\{f \geqslant e\}), essinf_m f\}$$

where $essinf_m f = \sup\{t \in [0,\infty] | m(\{f \geqslant t\}) = m(X)\}$.

Remark 2.12 [24]. Restricting now to the unit interval $[0,1]$ we shall consider functions $f \in \mathcal{F}^{(X,\mathcal{A})}$ satisfying $Ran(f) \subseteq [0,1]$ (in which case we shall write shortly $f \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$). Observe that, in this case, universal integrals are restricted to the class $\mathcal{D}_{[0,1]} = \bigcup_{(X,\mathcal{A}) \in \mathcal{S}} \mathcal{M}_1^{(X,\mathcal{A})} \times \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$.

Definition 2.13 [24]. Assume that $\circledast:[0,1]^2 \to [0,1]$ is a semicopula or a conjunctor (i.e., a binary operation $\circledast$ which is non-decreasing in both components, has 1 as neutral element and satisfies $a \circledast b \leqslant \min\{a,b\}$ for all $(a,b) \in [0,1]^2$, see [6,13]). The smallest universal integral $\mathbf{I}_{\circledast}$ on the $[0,1]$ scale related to the semicopula $\circledast$ is given by

$$\mathbf{I}_{\circledast}(m,f) = \sup\{t \circledast m(\{f \geqslant t\}) | t \in [0,1])\}.$$

This type of integral was called seminormed integral in [41]. Also, for a fixed strict t-norm T, the corresponding universal integral $\mathbf{I}_T$ is the so-called Sugeno–Weber integral [46].

Before starting our main results, we need the following definition:

Definition 2.14. Functions $f, g : X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0,$$

and f and g are said to be countermonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \leqslant 0.$$

The comonotonicity of functions f and g is equivalent to the nonexistence of points $x, y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$, or $f(x) > f(y)$ and $g(x) < g(y)$. Similarly, if f and g are countermonotone then $f(x) < f(y)$ and $g(x) < g(y)$ ($f(x) > f(y)$ and $g(x) > g(y)$) cannot happen. Observe that the concept of comonotonicity was first introduced in [10].

Now, our results can be stated as follows.

3. Main results

This section provides several type inequalities for universal integral.

3.1. Hardy's inequality

Before stating Hardy's inequality for universal integral, we need a lemma.

Lemma 3.1. *Let $H: [0,\infty)^n \to [0,\infty)$ be a continuous and non-decreasing n-place function, and let $\otimes: [0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$. Fix a $s \in (0,\infty)$ and suppose that H satisfies*

$$(H(p_1^s, p_2^s, \ldots, p_n^s) \otimes c) \geqslant \begin{pmatrix} H[(p_1 \otimes c)^s, p_2^s, p_3^s, \ldots, p_n^s] \\ \vee H[p_1^s, (p_2 \otimes c)^s, p_3^s, \ldots, p_n^s] \\ \vee \ldots \vee H[p_1^s, p_2^s, p_3^s, \ldots, (p_n \otimes c)^s] \end{pmatrix}$$

for all $p_1, p_2, \ldots, p_n$, $c \in [0,\infty)$. Then for any comonotone system $f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X,\mathcal{A})}$ and a monotone measure $m \in \mathcal{M}^{(X,\mathcal{A})}$ such that $a \otimes m(X) \leqslant a$ for all a and $\mathbf{I}_{\otimes}(m, f_i) < \infty$ for all $i = 1, 2, \ldots, n$, it holds

$$\mathbf{I}_{\otimes}(m, H(f_1^s, f_2^s, \ldots, f_n^s)) \geqslant H[\mathbf{I}_{\otimes}^s(m, f_1), \mathbf{I}_{\otimes}^s(m, f_2), \ldots, \mathbf{I}_{\otimes}^s(m, f_n)].$$

Proof. Let $e \in (0,\infty]$ be the neutral element of $\otimes$. If $\mathbf{I}_{\otimes}(m, f_i) = p_i < \infty$ for all $i = 1, 2, \ldots, n$, then for any $\varepsilon > 0$, there exist $p_{i(\varepsilon)}$ such that

$$m\left(\left\{f_i \geqslant p_{i(\varepsilon)}\right\}\right) = m\left(\left\{f_i^s \geqslant p_{i(\varepsilon)}^s\right\}\right) = M_i,$$

where $(p_{i(\varepsilon)} \otimes M_i)^s \geqslant (p_i - \varepsilon)^s$ for all $0 < s < \infty$. Since H is a non-decreasing function, the monotonicity of m and the comonotonicity of $f_1, f_2, \ldots, f_n$ imply that

$$m\left(\left\{H(f_1^s, f_2^s \ldots, f_n^s) \geqslant H[(p_{1(\varepsilon)})^s, (p_{2(\varepsilon)})^s \ldots, (p_{n(\varepsilon)})^s]\right\}\right)$$
$$\geqslant M_1 \wedge M_2 \wedge \ldots \wedge M_n.$$

Hence

$$\begin{aligned} \mathbf{I}_{\otimes}(m, H(f_1^s, f_2^s, \ldots, f_n^s)) &= \sup\{t \otimes m(\{H(f_1^s, f_2^s, \ldots, f_n^s) \geqslant t\}) | t \in (0,\infty])\} \\ &\geqslant (H((p_{1(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s) \otimes (M_1 \wedge M_2 \wedge \ldots \wedge M_n) \\ &= \begin{pmatrix} [H((p_{1(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s) \otimes M_1] \\ \wedge [H((p_{1(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s) \otimes M_2] \\ \wedge \ldots \wedge [H((p_{1(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s) \otimes M_n] \end{pmatrix} \\ &\geqslant \begin{pmatrix} H\left[(p_{1(\varepsilon)} \otimes M_1)^s, (p_{2(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s\right] \\ \wedge H[(p_{1(\varepsilon)})^s, (p_{2(\varepsilon)} \otimes M_2)^s, (p_{3(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s] \\ \wedge \ldots \wedge H[(p_{1(\varepsilon)})^s, (p_{2(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)} \otimes M_n)^s] \end{pmatrix} \\ &\geqslant \begin{pmatrix} H[(p_1 - \varepsilon)^s, (p_{2(\varepsilon)})^s, (p_{3(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s] \\ \wedge H[(p_{1(\varepsilon)})^s, (p_2 - \varepsilon)^s, (p_{3(\varepsilon)})^s, \ldots, (p_{n(\varepsilon)})^s] \\ \wedge \ldots \wedge H[(p_{1(\varepsilon)})^s, (p_{2(\varepsilon)})^s, (p_{3(\varepsilon)})^s, \ldots, (p_n - \varepsilon)^s] \end{pmatrix} \\ &\geqslant H[(p_1 - \varepsilon)^s, (p_2 - \varepsilon)^s, (p_3 - \varepsilon)^s, \ldots, (p_n - \varepsilon)^s], \end{aligned}$$

whence $\mathbf{I}_\otimes(m, H(f_1^s, f_2^s, \ldots, f_n^s)) = \sup\{t \otimes m(\{H(f_1^s, f_2^s, \ldots, f_n^s) \geqslant t\}) | t \in (0, \infty])\} \geqslant H(p_1^s, p_2^s \ldots, p_n^s)$ follows from the continuity of H and the arbitrariness of ε. And the theorem is proved. □

Lemma 3.1 helps us to reach the following result.

Theorem 3.2 (*Hardy type inequality for universal integral*). *Let $H{:}[0,\infty)^n \to [0,\infty)$ be a continuous and non-decreasing n-place function, and let $\otimes{:}[0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$. Fix a $s \in (0,\infty)$ and suppose that H satisfies*

$$\left(H(p_1^s, p_2^s, \ldots, p_n^s) \otimes c\right) \geqslant \begin{pmatrix} H\left[(p_1 \otimes c)^s, p_2^s, p_3^s, \ldots, p_n^s\right] \\ \vee H\left[p_1^s, (p_2 \otimes c)^s, p_3^s, \ldots, p_n^s\right] \\ \vee \ldots \vee H\left[p_1^s, p_2^s, p_3^s, \ldots, (p_n \otimes c)^s\right] \end{pmatrix}$$

for all $p_1, p_2, \ldots, p_n, c \in [0,\infty)$. Then for any comonotone system $f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X,\mathcal{A})}$ and a monotone measure $m \in \mathcal{M}^{(X,\mathcal{A})}$ such that $a \otimes m(X) \leqslant a$ for all a, $\mathbf{I}_\otimes(m,f_i) < \infty$ and $F_i(t) = \sup\{t \otimes m([0,x] \cap \{f_i \geqslant t\}) \,|\, t \in (0,\infty])\}, i = 1, 2, \ldots, n$, $x > 0$, it holds

$$\mathbf{I}_\otimes\left(m, \frac{\mathbf{I}_\otimes(m, H(f_1^s, f_2^s, \ldots, f_n^s))}{x^s}\right) \geqslant \mathbf{I}_\otimes\left(m, \frac{H(F_1^s(x), F_2^s(x), \ldots, F_n^s(x))}{x^s}\right).$$

Proof. By Lemma 3.1, we have

$$\mathbf{K} = \mathbf{I}_\otimes(m, H(f_1^s, f_2^s, \ldots, f_n^s)) \geqslant H\left[\mathbf{I}_\otimes^s(m,f_1), \mathbf{I}_\otimes^s(m,f_2), \ldots, \mathbf{I}_\otimes^s(m,f_n)\right].$$

Then

$$\begin{aligned}\mathbf{K} &\geqslant H\left[\mathbf{I}_\otimes^s(m,f_1), \mathbf{I}_\otimes^s(m,f_2), \ldots, \mathbf{I}_\otimes^s(m,f_n)\right] \\ &\geqslant H\left(F_1^s(x), F_2^s(x), \ldots, F_n^s(x)\right).\end{aligned}$$

Thus

$$\mathbf{I}_\otimes\left(m, \frac{\mathbf{K}}{x^s}\right) \geqslant \mathbf{I}_\otimes\left(m, \frac{H(F_1^s(x), F_2^s(x), \ldots, F_n^s(x))}{x^s}\right),$$

and the proof is completed. □

Corollary 3.3. *Let $f, g \in \mathcal{F}^{(X,\mathcal{A})}$ be two comonotone measurable functions and $\otimes{:}[0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $a \otimes m(X) \leqslant a$ for all a, $\mathbf{I}_\otimes(m,f_i) < \infty$ and $F_i(t) = \sup_{t\in(0,\infty]}\{t \otimes m([0,x] \cap \{f_i \geqslant t\})\}$, $i = 1, 2$ and $x > 0$. Let $\star{:}\ [0,\infty)^2 \to [0,\infty)$ be continuous and non-decreasing in both argument. If*

$$\left((p_1^s \star p_2^s) \otimes c\right) \geqslant \left[(p_1 \otimes c)^s \star p_2^s\right] \vee \left[p_1^s \star (p_2 \otimes c)^s\right],$$

then the inequality

$$\mathbf{I}_\otimes\left(m, \frac{\mathbf{I}_\otimes(m, (f_1^s \star f_2^s))}{x^s}\right) \geqslant \mathbf{I}_\otimes\left(m, \frac{(F_1^s(x) \star F_2^s(x))}{x^s}\right)$$

holds for all $0 < s < \infty$.

Notice that when working on [0,1] in Lemma 3.1 and Theorem 3.2, we mostly deal with $e = 1$, then $\otimes = \circledast$is semicopula (t-seminorm) and the following results hold:

Corollary 3.4 [5]. *Let $H{:}[0,1]^n \to [0,1]$ be a continuous and non-decreasing n-place function. Fix a $s \in (0,\infty)$ and suppose that semicopula $\circledast$ satisfies*

$$\left(H(p_1^s, p_2^s \ldots, p_n^s) \circledast c\right) \geqslant \begin{pmatrix} H\left[(p_1 \circledast c)^s, p_2^s, p_3^s, \ldots, p_n^s\right] \\ \vee H\left[p_1^s, (p_2 \circledast c)^s, p_3^s, \ldots, p_n^s\right] \\ \vee \ldots \vee H\left[p_1^s, p_2^s, p_3^s, \ldots, (p_n \circledast c)^s\right] \end{pmatrix} \quad (1)$$

for all $p_1, p_2, \ldots, p_n, c \in [0,1]$. Then for any comonotone system $f_1, f_2, \ldots, f_n \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ and a monotone measure $m \in \mathcal{M}_1^{(X,\mathcal{A})}$, it holds

$$\mathbf{I}_\circledast(m, H(f_1^s, f_2^s, \ldots, f_n^s)) \geqslant H\left[\mathbf{I}_\circledast^s(m,f_1), \mathbf{I}_\circledast^s(m,f_2), \ldots, \mathbf{I}_\circledast^s(m,f_n)\right].$$

Remark 3.5.

(I) Let $s = 1$, $n = 2$ in Corollary 3.4, then we get the Chebyshev type inequality for seminormed integral which obtained by Ouyang and Mesiar [33].
(II) For $s = 1$, $n = 2$, we can use an example in [33] to show that the condition (1) in Corollary 3.4 (and thus in Lemma 3.1 and Theorem 3.2) cannot be abandoned, and so we omit it here.

Corollary 3.6. *Let $H{:}[0,1]^n \to [0,1]$ be a continuous and non-decreasing n-place function. Fix a $s \in (0,\infty)$ and suppose that the semicopula $\circledast$ satisfies*

$$\left(H(p_1^s, p_2^s \ldots, p_n^s) \circledast c\right) \geqslant \begin{pmatrix} H[(p_1 \circledast c)^s, p_2^s, p_3^s, \ldots, p_n^s] \\ \vee H\left[p_1^s, (p_2 \circledast c)^s, p_3^s, \ldots, p_n^s\right] \\ \vee \ldots \vee H\left[p_1^s, p_2^s, p_3^s, \ldots, (p_n \circledast c)^s\right] \end{pmatrix}$$

for all $p_1, p_2, \ldots, p_n, c \in [0,1]$. Then for any comonotone system $f_1, f_2, \ldots, f_n \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ and a monotone measure $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ and $F_i(t) = \sup\{t \circledast m([0,x] \cap \{f_i \geqslant t\}) | t \in (0,1])\}$, $i = 1, 2, \ldots n$, $0 < x \leqslant 1$, it holds

$$\mathbf{I}_\circledast\left(m, \frac{\mathbf{I}_\circledast(m, H(f_1^s, f_2^s, \ldots, f_n^s))}{x^s}\right) \geqslant \mathbf{I}_\circledast\left(m, \frac{H(F_1^s(x), F_2^s(x), \ldots, F_n^s(x))}{x^s}\right).$$

Remark 3.7. If m is the Lebesgue measure on $\mathbb{R}$ in Corollary 3.6, then we have the Hardy type inequality for seminormed integral which obtained by Agahi and Yaghoobi [5].

3.2. Berwald's inequality

Theorem 3.8 (*Berwald type inequality for universal integral*). *Let $r, s \in (0,\infty)$ and $f \in \mathcal{F}^{([a,b],\mathcal{A})}$ be a concave function. If $\otimes{:}[0,\infty]^2 \to [0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$, then for any monotone measure $m \in \mathcal{M}^{(X,\mathcal{A})}$, we have*

(a) if $f(a) < f(b)$, then

$$\mathbf{I}_\otimes^{\frac{1}{r}}(m, f^r) \geqslant \left[\begin{matrix} \left(\dfrac{(1+s)^{\frac{1}{s}}(b-a)^{\frac{1}{r}-\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\mathbf{I}_\otimes^{\frac{1}{s}}(m, f^s)\right)^r \otimes \\ m\left([a,b] \cap \left\{x | x \geqslant \dfrac{\frac{(b-a)^{\frac{r+1}{r}-\frac{1}{s}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\mathbf{I}_\otimes^{\frac{1}{s}}(m,f^s) + af(b) - bf(a)}{f(b)-f(a)}\right\}\right) \end{matrix}\right]^{\frac{1}{r}}.$$

(b) If $f(a) = f(b)$, then

$$\mathbf{I}_\otimes^{\frac{1}{r}}(m, f^r) \geqslant \mathbf{I}_\otimes^{\frac{1}{r}}(m, f^r(a)).$$

(c) If $f(a) > f(b)$, then

$$\mathbf{I}_\otimes^{\frac{1}{r}}(m, f^r) \geqslant \left[\begin{matrix} \left(\dfrac{(1+s)^{\frac{1}{s}}(b-a)^{\frac{1}{r}-\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\mathbf{I}_\otimes^{\frac{1}{s}}(m, f^s)\right)^r \otimes \\ m\left([a,b] \cap \left\{x | x \leqslant \dfrac{\frac{(b-a)^{\frac{r+1}{r}-\frac{1}{s}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\mathbf{I}_\otimes^{\frac{1}{s}}(m,f^s) + af(b) - bf(a)}{f(b)-f(a)}\right\}\right) \end{matrix}\right]^{\frac{1}{r}}.$$

Proof. Let $r,\ s \in (0,\infty)$ and $\mathbf{I}_{\otimes}(m,f^s) = t$. Since $f:[a,b] \to [0,\infty)$ is a concave function, for $x \in [a,b]$ we have

$$f(x) = f\left(\left(1-\frac{x-a}{b-a}\right)a + \left(\frac{x-a}{b-a}\right)b\right)$$
$$\geqslant \left(1-\frac{x-a}{b-a}\right)f(a) + \left(\frac{x-a}{b-a}\right)f(b) = h(x).$$

(a) If $f(a) < f(b)$, then

$$\begin{aligned}
\mathbf{I}_{\otimes}^{\frac{1}{r}}(m,f^r) &\geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(m,h^r) \\
&= \left[\bigvee_{\alpha>0}\left(\alpha \otimes m\left([a,b]\cap\left\{\left[\left(1-\tfrac{x-a}{b-a}\right)f(a)+\left(\tfrac{x-a}{b-a}\right)f(b)\right] \geqslant \alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}} \\
&= \left[\bigvee_{\alpha>0}\left(\alpha \otimes m\left([a,b]\cap\left\{x \mid x \geqslant \tfrac{\alpha^{\frac{1}{r}}(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\right)\right]^{\frac{1}{r}} \\
&\geqslant \left[\begin{array}{c} \left(\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}\right)^{r} \otimes \\ m\left([a,b]\cap\left\{x \mid x \geqslant \frac{\left(\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)\right)}{f(b)-f(a)}\right\}\right)\end{array}\right]^{\frac{1}{r}}.
\end{aligned}$$

(b) If $f(a) = f(b)$, then $h(x) = f(a)$. Thus

$$\mathbf{I}_{\otimes}^{\frac{1}{r}}(m,f^r) \geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(m,h^r) = \mathbf{I}_{\otimes}^{\frac{1}{r}}(m,f^r(a)).$$

(c) If $f(a) > f(b)$, then

$$\begin{aligned}
\mathbf{I}_{\otimes}^{\frac{1}{r}}(m,f^r) &\geqslant \mathbf{I}_{\otimes}^{\frac{1}{r}}(m,h^r) \\
&= \left[\bigvee_{\alpha>0}\left(\alpha \otimes m\left([a,b]\cap\left\{\left[\left(1-\tfrac{x-a}{b-a}\right)f(a)+\left(\tfrac{x-a}{b-a}\right)f(b)\right] \geqslant \alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}} \\
&= \left[\bigvee_{\alpha>0}\left(\alpha \otimes m\left([a,b]\cap\left\{x \mid x \leqslant \tfrac{\alpha^{\frac{1}{r}}(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\right)\right]^{\frac{1}{r}} \\
&\geqslant \left[\begin{array}{c} \left(\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}\right)^{r} \otimes \\ m\left([a,b]\cap\left\{x \mid x \leqslant \frac{\left(\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)\right)}{f(b)-f(a)}\right\}\right)\end{array}\right]^{\frac{1}{r}},
\end{aligned}$$

and the proof is completed. □

Remark 3.9.

(I) If $\otimes$ is minimum in Theorem 3.8, then we obtain the Berwald type for Sugeno integral. Specially, when m is the Lebesgue measure on $\mathbb{R}$, then we have the results of [2].
(II) If $\otimes$ is the standard product in Theorem 3.8, then we have the Berwald type inequality for Shilkret integral.
(III) When working on $[0,1]$ in Theorem 3.8, then we mostly deal with $e = 1$, then $\otimes = \circledast$ is semicopula (t-seminorm). Then we have a Berwald type inequality for seminormed integral.

3.3. Barnes–Godunova–Levin's inequality

Theorem 3.10 (*Barnes–Godunova–Levin type inequality for universal integral*). *Let $p,\ q \in (0,\infty)$ and $f,g \in \mathcal{F}^{([a,b],\mathcal{A})}$ be two concave functions. Let $\otimes:[0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure. If the binary operation $\star$: $[0,\infty)^2 \to [0,\infty)$ is continuous and non-decreasing in both arguments, then*

(a) if $f(a) < f(b)$ and $g(a) < g(b)$, then

$$\left(\left[\mathbf{I}_{\otimes}^{\frac{1}{p}}(m,f^p)\star\mathbf{I}_{\otimes}^{\frac{1}{q}}(m,g^q)\right]\otimes\left(\begin{array}{l} m\left([a,b]\cap\left\{x \geqslant \frac{(b-a)\mathbf{I}_{\otimes}^{\frac{1}{p}}(m,f^p)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\wedge \\ m\left([a,b]\cap\left\{x \geqslant \frac{(b-a)\mathbf{I}_{\otimes}^{\frac{1}{q}}(m,g^q)+ag(b)-bg(a)}{g(b)-g(a)}\right\}\right)\end{array}\right)\right)$$
$$\leqslant \mathbf{I}_{\otimes}(m,f\star g).$$

(b) If $f(a) = f(b)$ and $g(a) = g(b)$, then

$$\mathbf{I}_{\otimes}(m,f(a)\star g(a)) \leqslant \mathbf{I}_{\otimes}(m,f\star g).$$

(c) If $f(a) > f(b)$ and $g(a) > g(b)$, then

$$\left(\left[\mathbf{I}_{\otimes}^{\frac{1}{p}}(m,f^p)\star\mathbf{I}_{\otimes}^{\frac{1}{q}}(m,g^q)\right]\otimes\left(\begin{array}{l} m\left([a,b]\cap\left\{x \leqslant \frac{(b-a)\mathbf{I}_{\otimes}^{\frac{1}{p}}(m,f^p)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\wedge \\ m\left([a,b]\cap\left\{x \leqslant \frac{(b-a)\mathbf{I}_{\otimes}^{\frac{1}{q}}(m,g^q)+ag(b)-bg(a)}{g(b)-g(a)}\right\}\right)\end{array}\right)\right)$$
$$\leqslant \mathbf{I}_{\otimes}(m,f\star g).$$

Proof. Let $p,q \in (0,\infty), \mathbf{I}_{\otimes}^{\frac{1}{p}}(m,f^p) = t_1$ and $\mathbf{I}_{\otimes}^{\frac{1}{q}}(m,g^q) = t_2$. Since f, $g:[a,b] \to [0,\infty)$ are two concave functions, for $x \in [a,b]$ we have

$$f(x) = f\left(\left(1-\frac{x-a}{b-a}\right)\cdot a + \left(\frac{x-a}{b-a}\right)\cdot b\right) \geqslant \left(1-\frac{x-a}{b-a}\right)\cdot f(a) + \left(\frac{x-a}{b-a}\right)\cdot f(b) = h_1(x),$$

$$g(x) = g\left(\left(1-\frac{x-a}{b-a}\right)\cdot a + \left(\frac{x-a}{b-a}\right)\cdot b\right) \geqslant \left(1-\frac{x-a}{b-a}\right)\cdot g(a) + \left(\frac{x-a}{b-a}\right)\cdot g(b) = h_2(x).$$

We will prove (a) and (b), the other case is similar.

(a) If $f(a) < f(b)$ and $g(a) < g(b)$, then by the comonotonicity of h_1 and h_2, we have

$$\begin{aligned}
\mathbf{I}_{\otimes}(m,f\star g) &\geqslant \mathbf{I}_{\otimes}(m,h_1\star h_2) = \bigvee_{\alpha>0}(\alpha\otimes m([a,b]\cap\{h_1(x)\star h_2(x) \geqslant \alpha\})) \\
&\geqslant (t_1\star t_2)\otimes m([a,b]\cap\{h_1(x)\star h_2(x) \geqslant t_1\star t_2\}) \\
&\geqslant (t_1\star t_2)\otimes (m([a,b]\cap\{h_1(x) \geqslant t_1\})\wedge m([a,b]\cap\{h_2(x) \geqslant t_2\})) \\
&= (t_1\star t_2)\otimes\left(\begin{array}{l} m\left([a,b]\cap\left\{\left(1-\frac{x-a}{b-a}\right)\cdot f(a)+\left(\frac{x-a}{b-a}\right)\cdot f(b) \geqslant t_1\right\}\right) \\ \wedge m\left([a,b]\cap\left\{\left(1-\frac{x-a}{b-a}\right)\cdot g(a)+\left(\frac{x-a}{b-a}\right)\cdot g(b) \geqslant t_2\right\}\right)\end{array}\right) \\
&= (t_1\star t_2)\otimes\left(\begin{array}{l} m\left([a,b]\cap\left\{x \geqslant \frac{t_1(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right) \\ \wedge m\left([a,b]\cap\left\{x \geqslant \frac{t_2(b-a)+ag(b)-bg(a)}{g(b)-g(a)}\right\}\right)\end{array}\right),
\end{aligned}$$

(b) If $f(a) = f(b)$ and $g(a) = g(b)$, then $h_1(x) = f(a)$ and $h_2(x) = g(a)$. Thus, we have

$$\mathbf{I}_{\otimes}(m,f\star g) \geqslant \mathbf{I}_{\otimes}(m,h_1\star h_2) = \mathbf{I}_{\otimes}(m,f(a)\star g(a)),$$

and the proof is completed. □

Remark 3.11.

(I) If $\otimes = \wedge$ and $\star$ is the standard product in Theorem 3.10, then we obtain an inequality related to Barnes–Godunova–Levin type for Sugeno integral which obtained by Agahi et al. [1].
(II) If $\otimes$ is the standard product in Theorem 3.10, then we have the Barnes–Godunova–Levin type inequality for Shilkret integral.
(III) When working on $[0,1]$ in Theorem 3.10, then we mostly deal with $e = 1$, then $\otimes = \circledast$ is semicopula (t-seminorm). Then we have a Barnes–Godunova–Levin type inequality for seminormed integral.

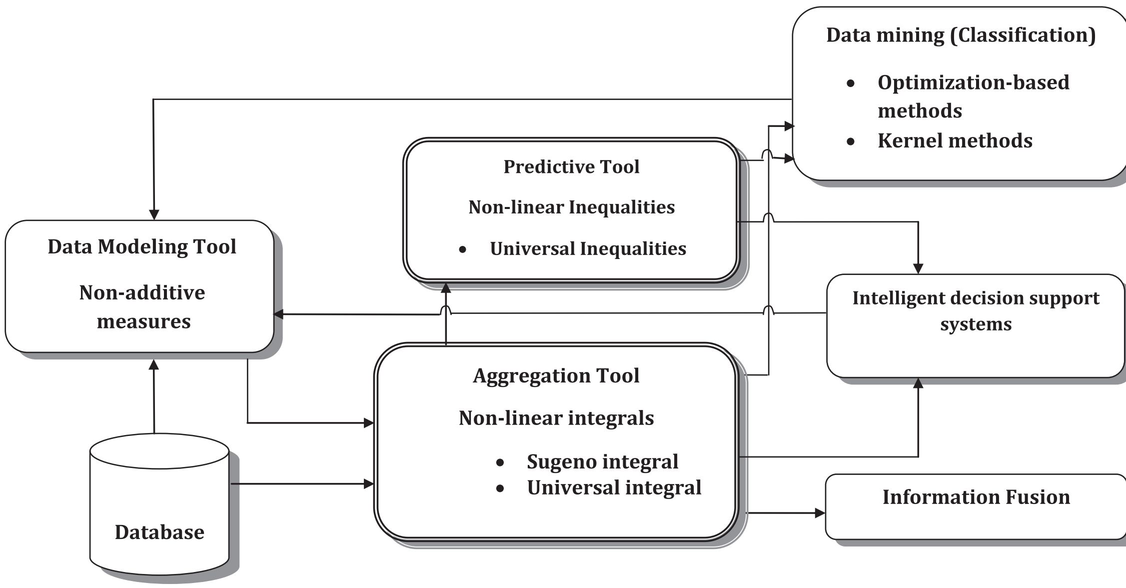

Fig. 1. Applications of non-additive integrals inequalities in non-linear systems.

3.4. Markov's inequality

Theorem 3.12 (*Markov type inequality for universal integral*). *Let $H:[0,\infty)^n \to [0,\infty)$ be a continuous and non-decreasing n-place function. Let $\varphi:[0,\infty) \to (0,\infty)$ be a non-decreasing function. If $\otimes$: $[0,\infty]^2 \to [0,\infty]$ is a pseudo-multiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$, then for any comonotone system $f_1, f_2, \ldots, f_n \in \mathcal{F}^{(X,\mathcal{A})}$ and a monotone measure $m \in \mathcal{M}^{(X,\mathcal{A})}$, it holds*

$$\mathbf{I}_{\otimes}(m, H(\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n))) \geqslant [H(\varphi(c_1), \varphi(c_2) \ldots, \varphi(c_n))] \otimes (M_1 \wedge M_2 \wedge \ldots \wedge M_n),$$

where $m(\{f_i \geqslant c_i\}) = M_i$, $i = 1, 2, \ldots, n$.

Proof. Let $m(\{f_i \geqslant c_i\}) = M_i$ for $i = 1, 2, \ldots, n$. Then the fact of

$$(\{\varphi(f_1) \geqslant \varphi(c_1)\} \cap \{\varphi(f_2) \geqslant \varphi(c_2)\} \cap \ldots \cap \{\{\varphi(f_n) \geqslant \varphi(c_n)\}\}) \subset \{H(\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n)) \geqslant H(\varphi(c_1), \varphi(c_2) \ldots, \varphi(c_n))\},$$

and the comonotonicity of $f_1, f_2, \ldots, f_n$ imply that

$$m(\{H(\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n)) \geqslant H(\varphi(c_1), \varphi(c_2) \ldots, \varphi(c_n))\}) \geqslant M_1 \wedge M_2 \wedge \cdots \wedge M_n.$$

Hence

$$\mathbf{I}_{\otimes}(m, H(\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n))) = \sup\{t \otimes m(\{H(\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_n)) \geqslant t\}) | t \in (0,\infty]\}\} \geqslant [H(\varphi(c_1), \varphi(c_2) \ldots, \varphi(c_n))] \otimes (M_1 \wedge M_2 \wedge \ldots \wedge M_n),$$

and the theorem is proved. □

Remark 3.13.

(I) If $\otimes = \wedge$ in Theorem 3.12, we obtain Markov type for Sugeno integral [37,39].
(II) If $\otimes$ is the standard product in Theorem 3.12, then we have Markov type inequality for Shilkret integral.
(III) When working on $[0,1]$ in Theorem 3.12, then we mostly deal with $e = 1$, then $\otimes = \circledast$ is semicopula (t-seminorm). Then we have Markov type inequality for seminormed integral.

4. Applications

Note that similarly as in the case of Lebesgue integral inequalities, also in non-additive integral case the inequalities are a theoretical tool for proving the existence, the uniqueness or some kind of optimality of the considered knowledge-based systems. In this section, we show some applications of our results in inteligent decision support systems, estimation and information fusion. In Fig. 1, we have depicted relationships between data processing tools, including non-additive measures, non-linear integrals and related inequalities, and knowledge-based systems, including data mining, intelligent decision support systems and information fusion. Some more details are given in subsequent subsections.

4.1. Intelligent decision support systems

Classical inequalities are based on the standard σ-additive Lebesgue measure (or probability measure) and they are at the heart of the mathematical analysis of many problems in the area of knowledge-based systems. However, additivity assumption seems to be illogical in many uncertain phenomena. In fact, many knowledge-based systems are built by means of non-classical techniques. In modeling human decision problems, additivity is rarely considered, as it excludes the possible interaction between single criteria. Thus, for human knowledge based systems such as intelligent decision support systems [48], we expect applications of non-additive integrals inequalities. Due to applications in multicriteria decision support, we focus on special non-additive integrals covered by the framework of universal integrals recently introduced in [24] and further discussed in [22]. Introduced non-additive integral inequalities generalize some recent results known from the literature and they promise to be useful in the comparison of effectiveness of single intelligent decision support systems.

4.2. Estimation

Note that classical integral inequalities mentioned above are frequently applied in estimation, approximation and convergence problems. As they are based on a classical Lebesgue measure, they fit to problems in areas where the additivity is a genuine property, such as engineering or natural sciences [27]. In fact, the applications of inequalities mentioned above are useful in practical problems by providing lower and upper bounds on the machine learning. For example, the need and importance of consideration of this kind of Markov inequality stemmed from several problems in engineering and informational sciences [18,19]. In fact, we use Markov's inequality to derive an upper bound on the difference between our estimated coefficients and the actual values in engineering and non-linear systems.

4.3. Information fusion

Information fusion is an especially difficult problem for non-linear systems. The integral inequalities are useful tools in information fusion and sensor networks. For example, Yang et al. [47] conducted comprehensive analytical and simulation studies on collaborative information coverage and object detection in wireless sensor networks by using Markov's inequality. Notice that the traditional tool of aggregation for information fusion is the weighted average method that is fundamentally a Lebesgue integral. It is considered that all attributes are non-interactive and, hence, their weighted effects are observed as additive ones. This assumption is often unrealistic in many applications [45]. Universal integrals can be used to solve these problems in information fusion. There are many successful works in this area. One of them is given by Keller et al. [20] which presents the application of Sugeno integral in image processing. So, our results are necessary when we use non-linear integrals.

5. Conclusion

We have introduced several inequalities including Hardy, Berwald, Barnes–Godunova–Levin, Markov and Chebyshev for universal integral on abstract spaces. We have also provided a comprehensive study of our results in intelligent systems. We would like to attract the readers' attention to a software R package called `kappalab` [16], which is available on the Comprehensive R Archive Network (http://cran.r-project.org/web/packages/kappalab/index.html). It allows the user to compute several non-additive integrals. In the future research, we will continue to explore other integral inequalities for non-additive measures and integrals.

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